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Schrödinger cat states in a Penning trap

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Abstract. The time-dependent Schrödinger equation of an ion moving in an asymmetric Penning trap is solved. Properties of the evolution of squeezed and Schrödinger cat states in the trap are studied. Analytic expressions for the energy expectation values, dispersions and correlations of the position and momentum operators of different macroscopic states, solutions of the time-dependent Schrödinger equation of the trap, are obtained. The probability densities in coordinate and momentum representations are also given. As an example, the behaviour of Schrödinger cat states of a proton moving in the trap is described numerically.

1. Introduction

Schrödinger cat states are quantum superpositions of macroscopically distinguishable states. They can be produced in quantum optical experiments, as pointed out by Yurke and Stoler [1]. From a theoretical point of view, they are even and odd coherent states; they were introduced by Man'ko [2] and studied by Castaños *et al* [3] for the case of a parametrically excited generalized harmonic oscillator.

Particle traps permit us to make extremely precise measurements and test the most basic postulates of modern physics. For example, the production of antihydrogen using ion traps has been discussed in connection with testing CPT symmetry and the weak equivalence principle for antiparticles [4]. Experiments have also been suggested to demonstrate that radiation appears in vacuum in an accelerated reference frame [5, 6]. Amongst the most interesting recent experimental results, we can mention the creation of Schrödinger cat states in different types of experiments [7]. Recently, Monroe *et al* [8] reported the generation of such a state at the single atom level in a coaxial resonator ion trap.

A relatively simple set-up is the Penning trap, in which charged particles are kept in the combination of a homogeneous magnetic field and an electrostatic quadrupole potential [9, 10]. This trap can be modelled by a two-mode symmetric harmonic oscillator in a rotating frame, a system equivalent to the generalized oscillator [11], which has been studied with the formalism of linear integrals of motion [12, 13]. It was recently shown that charged particles in a Penning trap with an asymmetric electrostatic field can be in squeezed states [14]; in the limit of an axially symmetric trap, the model is equivalent to the generalized oscillator considered in [11].

The aim of this paper is to study the Schrödinger cat states of a charged particle in a Penning trap, using the previously mentioned formalism [12, 14]. These states are superpositions of the time-dependent solutions of the Schrödinger equation that correspond to generalized coherent correlated states, which are also superpositions of energy eigenstates. We study their evolution under the influence of an axially asymmetric electrostatic field, and find the energy expectation values, and the probability densities, together with the coordinates and momentum dispersions; numerical results are presented for the physically realistic case of a proton in the trap. A similar problem for a Paul trap was recently analysed by Castaños *et al* [15].

This paper is organized as follows. In section 2 we describe the asymmetric Penning trap, find the general solution for the Schrödinger wave equation and calculate analytic expressions for the energy expectation values with respect to the generalized correlated states. In section 3 we study even and odd Schrödinger cat states in an asymmetric Penning trap; we calculate their energy expectation values, dispersions and correlations for coordinates and momenta, and probability density functions, both in coordinates and momenta. Numerical results are presented in section 4 for a proton moving in the trap. Finally, a summary of the results is given in section 5.

2. Invariants of motion and Penning trap

The dynamical system associated to any quadratic Hamiltonian with coordinates and momenta $\tilde{\boldsymbol{q}} = \{q_i\}$ and $\tilde{\boldsymbol{p}} = \{p_i\}$ admits a solution in the form of invariants of motion which have the commutation relations of creation and annihilation operators in the quantum version of the problem (here and in the following a tilde means the transpose of the matrix). Thus, the time invariant annihilation operator takes the form [16]:

$$\boldsymbol{A}_0(t) = \lambda_p(t)\boldsymbol{p} + \lambda_q(t)\boldsymbol{q} \tag{1}$$

where matrices λ_p and λ_q follow from the classical equations of motion [3]; here, $\boldsymbol{A}_0$ and $\boldsymbol{A}_0^\dagger$ depend explicitly on time but are constants of motion, they coincide with the two-dimensional harmonic oscillator annihilation and creation operators at time $t = 0$.

As a consequence, any quadratic Hamiltonian can be written in the form

$$H = -\frac{\hbar^2}{2}\,(\tilde{\boldsymbol{A}}_0 \quad \tilde{\boldsymbol{A}}_0^\dagger)\begin{pmatrix} \boldsymbol{K} & \boldsymbol{L}^* \\ \boldsymbol{L} & \boldsymbol{K}^* \end{pmatrix}\begin{pmatrix} \boldsymbol{A}_0 \\ \boldsymbol{A}_0^\dagger \end{pmatrix} \tag{2}$$

where $\boldsymbol{K}$ and $\boldsymbol{L}$ are complex, Hermitian matrices, respectively.

From the constants of motion it is possible to construct the solutions of the corresponding quantum problem, that is, solutions of the Schrödinger equation with the Hamiltonian equation (2). Indeed, the squeezed states are described in coordinate representation by the wavefunction [16]

$$\psi_\alpha(\boldsymbol{q}, t) = \hbar^{-1}(2\pi \det \lambda_p)^{-\frac{1}{2}} \exp\left\{-\frac{\alpha^\dagger\alpha}{2} + \frac{1}{2}\tilde{\alpha}\lambda_p^*\lambda_p^{-1}\alpha\right\}$$
$$\times \exp\left\{\hbar^{-1}\left(-\frac{\mathrm{i}}{2}\tilde{\boldsymbol{q}}\lambda_p^{-1}\lambda_q\boldsymbol{q} + \mathrm{i}\tilde{\alpha}\tilde{\lambda}_p^{-1}\boldsymbol{q}\right)\right\} \tag{3}$$

where $\tilde{\alpha} = \{\alpha_i\}$ is the multimode field amplitude.

Due to the symmetry between coordinates and momenta, it is clear that the wavefunction in the momentum representation can be obtained from the corresponding wavefunction in coordinate representation making the substitutions

$$\boldsymbol{q} \longrightarrow \boldsymbol{p}$$
$$\lambda_p \longrightarrow -\lambda_q .$$

As a consequence, wavefunctions with different spatial support also have different supports in momentum representations.

Recalling now that $\boldsymbol{A}_0|\alpha\rangle = \alpha|\alpha\rangle$, it is easy to obtain the expectation values for the Hamiltonian with respect to the multidimensional coherent states $|\alpha\rangle$:

$$\langle H\rangle_\alpha = -\frac{\hbar^2}{2}(\tilde{\alpha}\boldsymbol{K}\alpha + \alpha^\dagger\boldsymbol{K}^*\alpha^* + \mathrm{tr}(\boldsymbol{L}) + 2\alpha^\dagger\boldsymbol{L}\alpha). \tag{4}$$

Now, in the particular case of a charged particle (of mass m) in an asymmetric Penning trap, the Hamiltonian (discarding the harmonic movement with frequency ω_z along the z-axis) can be written in the form (2), with the matrices

$$\boldsymbol{K} = \quad \frac{1}{m}\lambda_q^*\lambda_q^\dagger + \frac{1}{2}\omega_c(\lambda_q^*\Sigma\lambda_p^\dagger - \lambda_p^*\Sigma\lambda_q^\dagger) + m\lambda_p^*\Omega^2\lambda_p^\dagger \tag{5}$$

$$\boldsymbol{L} = -\frac{1}{m}\lambda_q\lambda_q^\dagger + \frac{1}{2}\omega_c(\lambda_p\Sigma\lambda_q^\dagger - \lambda_q\Sigma\lambda_p^\dagger) - m\lambda_p\Omega^2\lambda_p^\dagger \tag{6}$$

where $\Omega = \mathrm{diag}(\omega_x, \omega_y)$, and Σ is the 2×2 symplectic metric. The frequencies ω_x and ω_y are defined by the expressions

$$\omega_{x,y}^2 = \tfrac{1}{4}\omega_c^2 - \tfrac{1}{2}(1 \pm D)\omega_z^2. \tag{7}$$

Here ω_c is the cyclotron frequency [10], and D is a parameter that measures the axial asymmetry [14].

Equation (2), for the asymmetric Penning trap, must be time independent; therefore the Hamiltonian itself is a constant of motion and its expectation value with respect to generalized correlated states is given by

$$\langle H\rangle_\alpha = \hbar\omega_x\left(r_1^2 + \frac{1}{2}\right) + \hbar\omega_y\left(r_2^2 + \frac{1}{2}\right) + \frac{\hbar\omega_c r_1 r_2}{2\sqrt{\omega_x\omega_y}}$$
$$\times\{(\omega_y - \omega_x)\sin(\phi_1 + \phi_2) + (\omega_x + \omega_y)\sin(\phi_2 - \phi_1)\} \tag{8}$$

where we used the polar form $\alpha_j = r_j\exp(\mathrm{i}\phi_j)$, with $j = 1, 2$ for the state parameters, and the values of the matrices $\boldsymbol{K}$ and $\boldsymbol{L}$, given by equations (5) and (6) for $t = 0$.

In the particular case of a charged particle in a Penning trap, the realistic values of the frequencies for an electron in the trap are such that $\omega_c \gg \omega_z \gg \omega_m$, where $\omega_m \simeq \omega_z^2/2\omega_c$ is the magnetron frequency (these three frequencies are in the ranges of GHz, MHz and kHz, respectively [10]). In this case, as shown in [14], the squeezing coefficients oscillate with a period of the order of the magnetron frequency ω_m, provided that $\omega_c \gg \omega_z \gg \omega_m$. This approximation is very good for an electron in the trap, but fails for more massive particles such as a proton or an ion. In the following, the general case will be studied without approximations.

3. Schrödinger cat states

The normalized coherent states $|\alpha\rangle$ are eigenstates of the annihilation operator a:

$$a|\alpha\rangle = \alpha|\alpha\rangle \tag{9}$$

where α is a complex number and $a^\dagger a|n\rangle = n|n\rangle$, with $n = 0, 1, 2, \ldots$. The even and odd coherent states $|\alpha_\pm\rangle$ are superpositions of the coherent states [2]

$$|\alpha_\pm\rangle = N_\pm(|\alpha\rangle \pm |-\alpha\rangle) \tag{10}$$

where the normalization constants have the form:

$$N_+ = \frac{e^{|\alpha|^2/2}}{2\sqrt{\cosh|\alpha|^2}} \qquad N_- = \frac{e^{|\alpha|^2/2}}{2\sqrt{\sinh|\alpha|^2}}. \tag{11}$$

The multimode generalization of the cat states is straightforward [17].

The density functions associated to these wavefunctions are

$$\rho_\alpha^{(\pm)}(\boldsymbol{q},t) = |N_\pm|^2\{|\psi_\alpha(\boldsymbol{q},t)|^2 + |\psi_{-\alpha}(\boldsymbol{q},t)|^2 \pm \psi_\alpha(\boldsymbol{q},t)\psi^*_{-\alpha}(\boldsymbol{q},t) \pm \psi_{-\alpha}(\boldsymbol{q},t)\psi^*_\alpha(\boldsymbol{q},t)\}. \tag{12}$$

To write down the explicit expression for the density, use equation (3) and the two-mode generalization of the normalization constant equation (11).

The corresponding expectation values of the Hamiltonian with respect to cat states are given by

$$\langle H\rangle_{\alpha\pm} = -\frac{\hbar^2}{2}\{\tilde{\alpha}\boldsymbol{K}\alpha + \alpha^\dagger\boldsymbol{K}^*\alpha^* + \mathrm{tr}(\boldsymbol{L}) + 2\Delta_\pm\alpha^\dagger\boldsymbol{L}\alpha\} \tag{13}$$

where $\Delta_\pm$ is defined as

$$\Delta_\pm = \begin{cases} \tanh(\alpha^\dagger\alpha) & \text{for even cats} \\ \coth(\alpha^\dagger\alpha) & \text{for odd cats.} \end{cases} \tag{14}$$

These expectation values of the energy of cat states are again time independent and are given by

$$\begin{aligned}\langle H\rangle_{\alpha\pm} = {}& \hbar\omega_x(\Delta_\pm r_1^2 + \tfrac{1}{2}) + \hbar\omega_y(\Delta_\pm r_2^2 + \tfrac{1}{2}) + \frac{\hbar\omega_c r_1 r_2}{2\sqrt{\omega_x\omega_y}} \\ & \times\{(\omega_y - \omega_x)\sin(\phi_1+\phi_2) + \Delta_\pm(\omega_x+\omega_y)\sin(\phi_2-\phi_1)\}\end{aligned} \tag{15}$$

where again the polar forms for α_i are used.

For the Schrödinger cat states, the expectation values of the coordinates and momentum states are obviously zero. As for the dispersions, they are given by the dispersion matrix

$$\boldsymbol{\sigma}^\pm(t) = \boldsymbol{\Sigma}\tilde{\boldsymbol{\Lambda}}\boldsymbol{\Sigma}\boldsymbol{\sigma}^\pm(0)\boldsymbol{\Sigma}\boldsymbol{\Lambda}\boldsymbol{\Sigma} \tag{16}$$

where $\boldsymbol{\Sigma}$ is the symplectic matrix, and $\boldsymbol{\sigma}^\pm(0)$ is the dispersion matrix of Schrödinger cat states constructed from standard two-mode coherent states; they are given by the expressions

$$(\sigma^\pm_{p_i p_j})(0) = \frac{\sqrt{\omega_{q_i}\omega_{q_j}}}{\omega_0}\left\{\mathrm{Re}\,[\alpha_i(-\alpha_j + \Delta_\pm\alpha_j^*)] + \frac{1}{2}\delta_{ij}\right\} \tag{17}$$

$$(\sigma^\pm_{p_i q_j})(0) = \sqrt{\frac{\omega_{q_i}}{\omega_{q_j}}}\,\mathrm{Im}\,[\alpha_i(\alpha_j + \Delta_\pm\alpha_j^*)] = (\sigma^\pm_{q_j p_i})(0) \tag{18}$$

$$(\sigma^\pm_{q_i q_j})(0) = \frac{\omega_0}{\sqrt{\omega_{q_i}\omega_{q_j}}}\left\{\mathrm{Re}\,[\alpha_i(\alpha_j + \Delta_\pm\alpha_j^*)] + \frac{1}{2}\delta_{ij}\right\} \tag{19}$$

with $\omega_{\mathrm{q}_1} = \omega_x$, $\omega_{\mathrm{q}_2} = \omega_y$ and

$$\omega_0^2 = \tfrac{1}{4}\omega_c^2 - \tfrac{1}{2}\omega_z^2. \tag{20}$$

4. Numerical results

We now study the case of a proton confined in a Penning trap. As typical parameters of the trap, we take the following values. For the electrostatic potential between the plates: $V_0 = 53.1$ V, for the characteristic length: $d = 0.112$ cm; and the constant magnetic field is taken as $B_0 = 50.5$ kg [10]. These parameters of the trap yield the cyclotronic, ω_c, the magnetronic, ω_m, and in direction z, ω_z, frequencies

$$\omega_c = 483.97 \text{ MHz} \qquad \omega_m = 4.17 \text{ MHz} \qquad \omega_z = 63.22 \text{ MHz}. \tag{21}$$

Next, the probability densities in coordinate space and dispersions of the positions and momenta are calculated for two cases: Schrödinger cat states and generalized correlated states of a proton moving in an asymmetric Penning trap with $D = 0.3$. In these calculations, we have used the following units:

$$[t] = 4.21 \times 10^{-9} \text{ s} \qquad [q] = 1.63 \times 10^{-6} \text{ cm} \qquad [p] = 6.48 \times 10^{-22} \text{ g cm s}^{-1}. \tag{22}$$

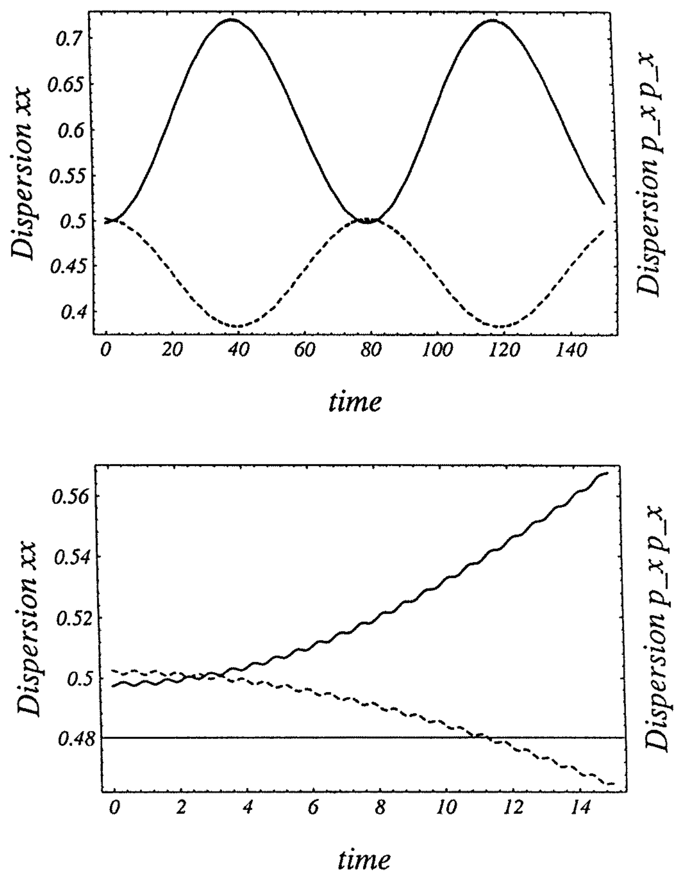

Figure 1. Quadrature dispersions for the generalized correlated state. The dispersions along the first direction in the coordinates and momentum are plotted, with broken and full curves, respectively. They are displayed for magnetronic and cyclotronic times. Hereafter, the scales are those given in section 4.

4.1. Correlated states

By means of expression (8), the expectation value of the Hamiltonian with respect to correlated states can be calculated. The minimum energy values for the set of parameter amplitudes $(\alpha_1, \alpha_2) = (r, r)$, $(r, 0)$, or $(0, r)$ are associated to the correlated vacuum state, whereas numerical calculations show that smaller energy values are obtained for correlated states with amplitude modes $\alpha_1 = \mathrm{i}r$ and $\alpha_2 = 0.3$, or vice versa, than the correlated vacuum state.

As discussed in [14], an asymmetry in the electrostatic field in the Penning trap produces a squeezed state for a particle moving in the trap. An analytic expression was obtained for the dispersion of the position and momenta of the particle in [14], and an approximate formula given for the case of an electron. Here we have calculated the dispersion numerically for a proton. The dispersions of the coordinate x and momentum p_x are shown in figure 1. The behaviour of the system is shown in the upper part for magnetronic times and in the lower part for cyclotronic times. The squeezing phenomenon in the position variable is clearly present and the dispersions have periodic variations. The dispersions in the other coordinates and momentum have the same qualitative form and will not be shown here.

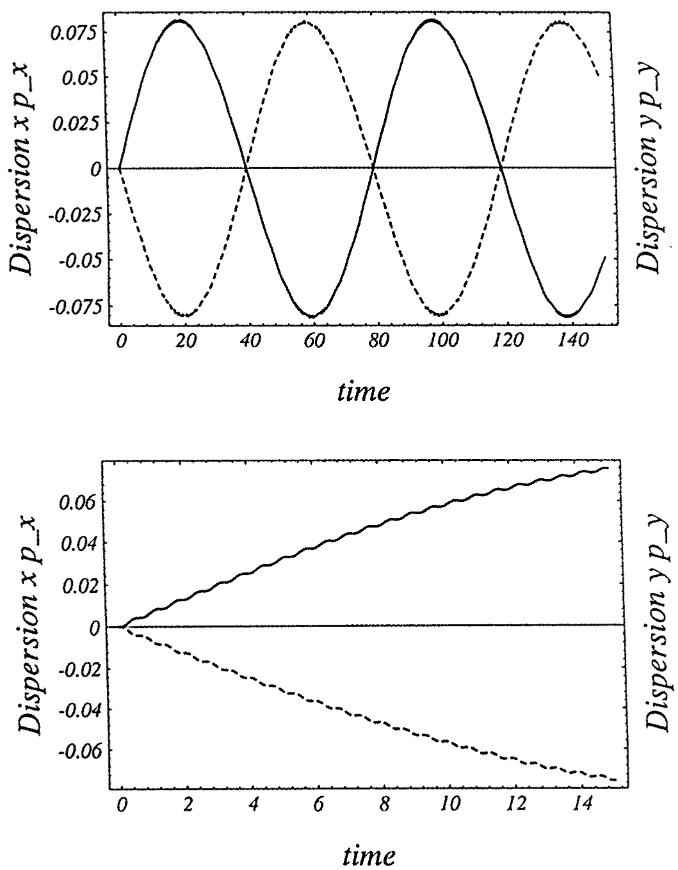

Figure 2. Correlation coefficients between momenta and positions in the x and y directions are plotted for the state of the proton moving in the asymmetric Penning trap.

The correlations of the positions and momenta are illustrated in figure 2. They also show a periodic behaviour around zero. It is worth noticing that the dispersions and correlations vary periodically over two superimposed timescales: one corresponding to the cyclotron frequency and the other to the magnetron frequency.

4.2. Schrödinger cat states

We now consider the even and odd Schrödinger cat states. In these cases the expectation value of the Hamiltonian is calculated through expression (15). For the same set of parameters used in section 4.1, the energy values are always smaller for the even Schrödinger cat state than for the odd case. However, for $\alpha_1 = \mathrm{i}r$ and $\alpha_2 = 0.3$, or vice versa, the energy values, around $r = 0.3$, are smaller for the odd states than for the even states, and besides have *lower* energies than the correlated vacuum state.

The quadrature dispersions for these states were calculated for the two-mode field amplitudes $\alpha_1 = \alpha_2 = \frac{1}{3}$ that fixes the initial positions and momenta. Again there is a variation of the dispersions of the position and momentum coordinates over the same two

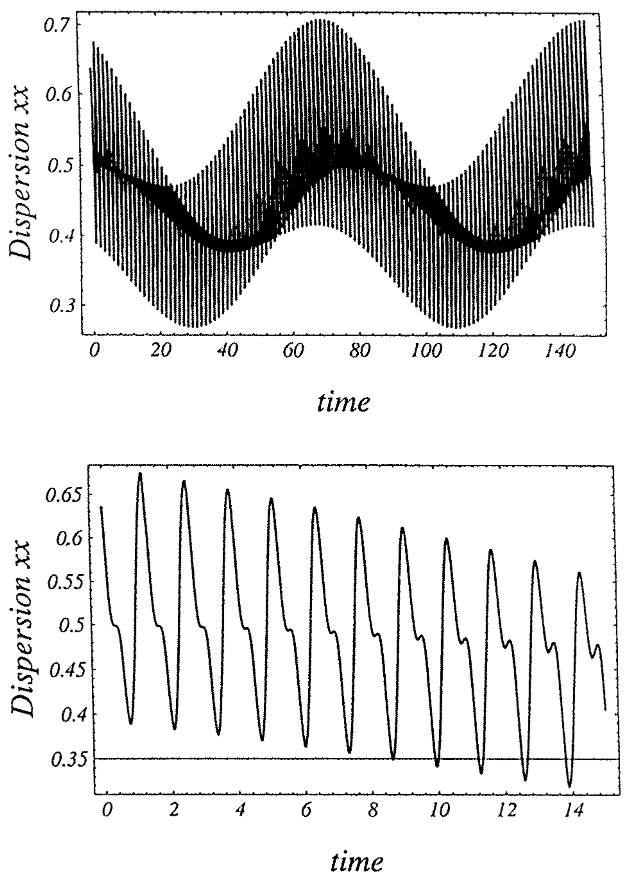

Figure 3. Quadrature dispersion in the x direction for the even Schrödinger, plotted for magnetronic and cyclotronic times. The parameters of the amplitudes of the state are $\alpha_1 = \alpha_2 = \frac{1}{3}$.

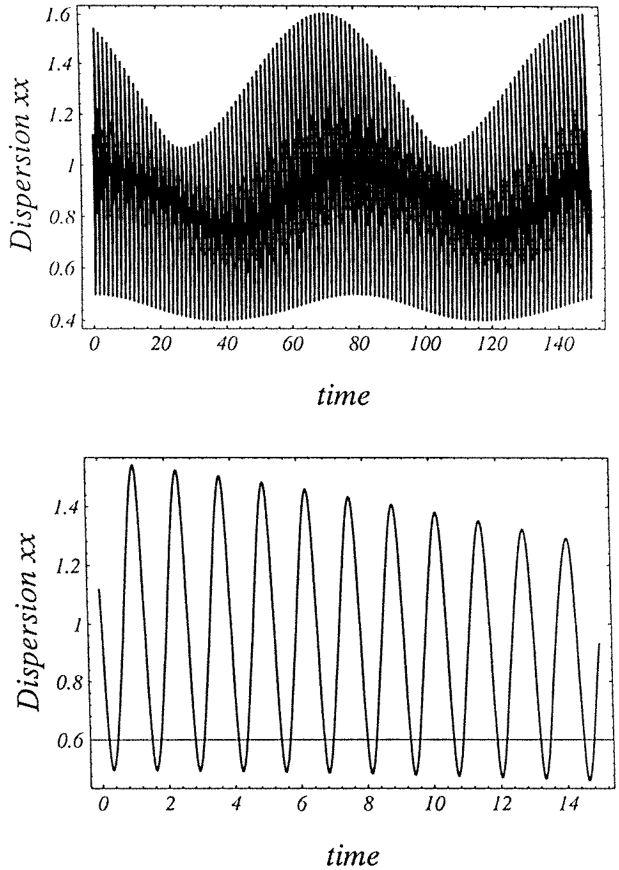

Figure 4. Quadrature dispersion in the x direction for the odd Schrödinger cat state, plotted for magnetronic and cyclotronic times. The parameters of the amplitudes of the state are $\alpha_1 = \alpha_2 = \frac{1}{3}$.

timescales as in the above case. The typical behaviour of the dispersions in the x coordinate are shown in figure 3 for the even state and in figure 4 for the odd case.

It is also interesting to display the evolution of the probability density in coordinate space. The cat states are characterized by a superposition of two well-localized probability densities with an interference term between them. This interference is most conspicuous for the even state, as it can be seen in figure 5: the two localized states move one in front of the other, with an interference that blows up periodically each time they come close together. This behaviour is also evident for the odd state, shown in figure 6; however, in this case the interference is destructive since the probablity density of this state has a node in the origin.

The above numerical results, with the given parameters, are quite representative of the general characteristics of the correlated and Schrödinger cat states of the motion of an ion moving in an asymmetric Penning trap. Further numerical explorations with different sets of parameters were performed, but they did not show any qualitatively different features; therefore, they are not presented here.

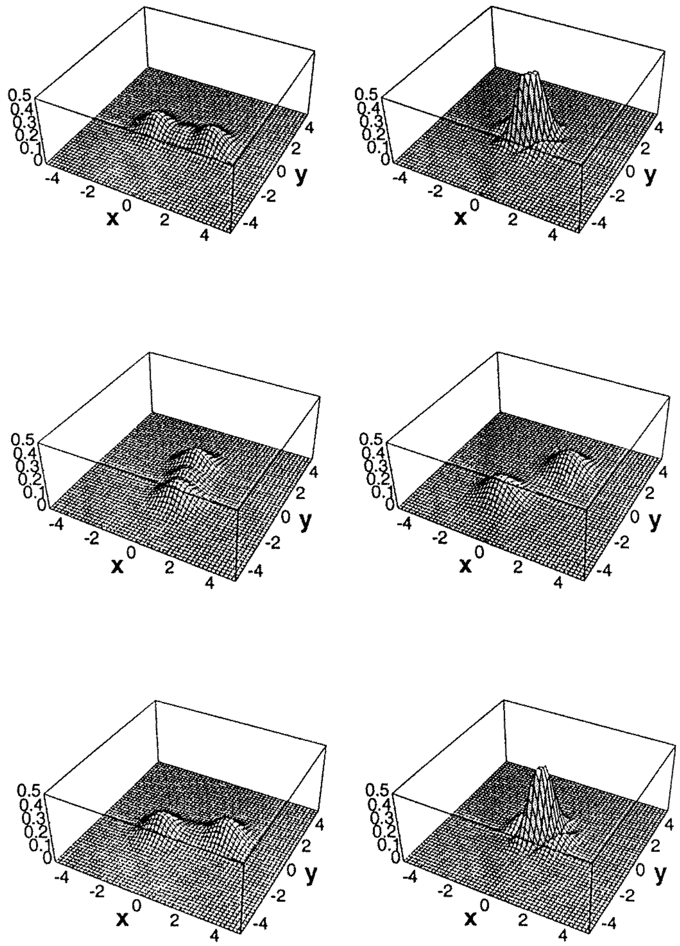

Figure 5. Evolution of the probability density in the configuration space of even Schrödinger cat states of the proton moving in the asymmetric Penning trap. This is plotted for the amplitudes $\alpha_1 = \alpha_2 = 1.0$; starting from the top to the bottom and from the right to the left with $t = 1$ and steps of one unit of time.

5. Summary

The present interest in ion traps is well justified by the possibility of making extremely precise measurements and test many of the fundamental concepts of modern physics. In this paper, we have shown how a Schrödinger cat state behaves in a Penning trap. For the expectation value of the energy, for the confinement of a proton, the asymmetry parameter D does not have an important role. Two-mode amplitudes can be found for which the energy values are smaller for correlated, even and odd states, than for the correlated vacuum state.

The key feature in the behaviour of an ion in a Penning trap is the fact that there are essentially two different timescales, corresponding to the magnetron (slow) and the cyclotron (fast) motion of the particle in the trap. Our numerical calculations show that for an even or odd superposition, the two states rotate around one another in a stable way, and that the frequency of this rotation corresponds to the magnetron frequency. For the even state, there is a very strong interference term which pops up periodically every time the two states get

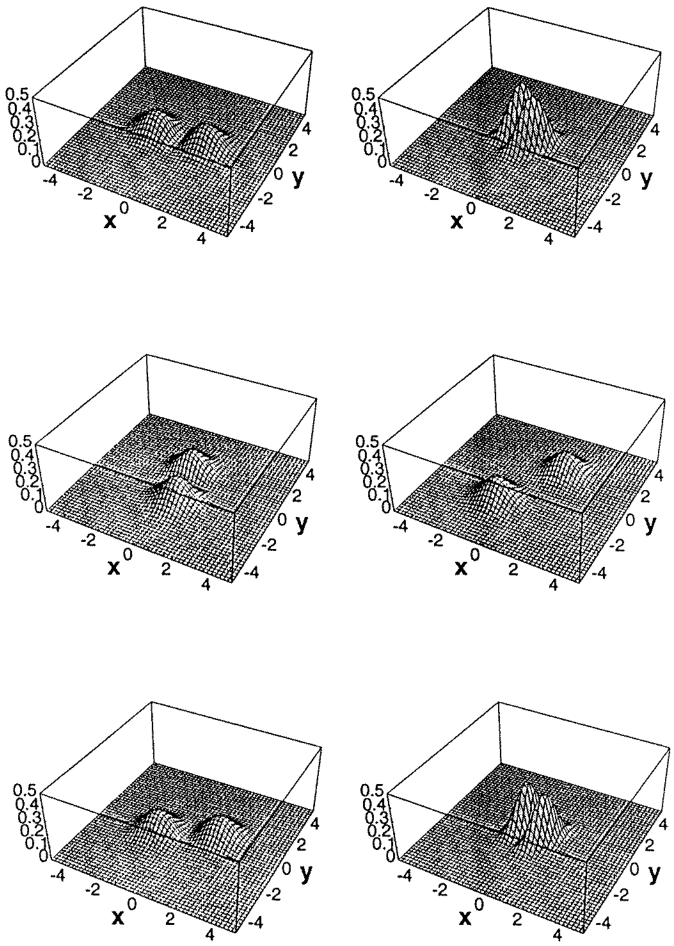

Figure 6. Evolution of the probability density in the configuration space of odd Schrödinger cat states of the proton moving in the asymmetric Penning trap, plotted for the amplitudes $\alpha_1 = \alpha_2 = 1.0$; starting from the top to the bottom and from the right to the left with $t = 1$ and steps of one unit of time.

close enough. We have also shown how the dispersions of these superposed states behave in time: the peculiar feature in this case is the squeezing over the two superimposed timescales mentioned above. The figures presented here show the essential behaviour. Other values of the parameters can be worked out, depending on the specific problem, but qualitative changes are not to be expected. In order to make a comparison with experimental results, one has to study the interaction of the Schrödinger cat states with an electromagnetic field; this is an open problem to which we expect the present work to make an initial contribution.

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